Homogeneous eqn with analytic coefficients

Defn: Analytic function

If g is a function defined on an interval I containing a point x_0, we say that g is analytic at x_0 if g can be expanded in a power series about x_0 which has positive radius of convergence. Thus g is analytic at x_0 if it can be represented in the form,

$$g(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k$$

where the c_k are constants and the series converges for $|x-x_0| < r_0$, $r_0 > 0$

Theorem: 12 (Existence Theorem for analytic coefficients)

Let x_0 be a real number and suppose that the coefficients $a_1, a_2, \dots a_n$ in $L(y) = y^{(n)} + a_1(x) y^{(n-1)} + \dots + a_n(x) y$. have convergent power series expansion in powers of $x-x_0$ on an interval $|x-x_0| < r_0$, $r_0 > 0$

If $\alpha_1, \alpha_2, \dots \alpha_n$ are any n constants there exists a soln φ of the problem with a power series expansion

$$\varphi(x) = \sum_{k=0}^{\infty} c_k (x-x_0)^k \rightarrow ①$$

convergent for $|x-x_0| < r_0$.

We have

$$k!\, c_k = \begin{cases} \alpha_{k+1} & (k=0,1,2 \dots n-1) \\ c_k & \text{for } k \geq n \end{cases}$$

may be computed in terms of $c_0, c_1, \dots c_{n-1}$ by substituting the series ① into $L(y)=0$.

Problem.

a) Find two linearly independent power series (in powers of x) of the eqn $y''-x^2y=0$

Soln:

Given eqn is $y''-x^2y=0 \rightarrow (1)$

Here $a_1(x)=0$, $a_2(x)=-x^2$ and a_1, a_2 are analytic for all real x.

Let $\varphi(x)=\sum_{k=0}^{\infty} c_k x^k$

$$\varphi'(x)=\sum_{k=1}^{\infty} k c_k x^{k-1} = \sum_{k=0}^{\infty} k c_k x^{k-1}$$

$$\varphi''(x)=\sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$ replacing k by $k+2$

$$\varphi''(x)=\sum_{k=0}^{\infty} (k+2)(k+1)c_{k+2}x^k$$

Sub in (1) we obtain

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=0}^{\infty}c_k x^{k+2}=0$$

put $k=k-2$ in second summation

$$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k-\sum_{k=2}^{\infty}c_{k-2}x^k=0$$

ie) $$(2)(1)c_2+(3)(2)c_3x+\left(\sum_{k=2}^{\infty}(k+2)(k+1)c_{k+2}-\sum_{k=2}^{\infty}c_{k-2}\right)x^k=0$$

ie) $$2c_2+6c_3x+\sum_{k=2}^{\infty}\left[(k+2)(k+1)c_{k+2}-c_{k-2}\right]x^k=0$$

This is true only if all the coefficients of the powers of x are zero.

$c_2=0$, $c_3=0$. $(k+2)(k+1)c_{k+2}=c_{k-2}$

$$\boxed{c_{k+2}=\frac{c_{k-2}}{(k+2)(k+1)}, \quad k=2,3,4\ldots}$$

for $k=2$, $c_4=\frac{c_0}{3.4}$

$k=3$, $c_5=\frac{c_1}{4.5}$

$k=4$, $\quad c_6 = \frac{c_2}{5\cdot 6} = 0 \qquad \because c_2 = 0$

$k=5$, $\quad c_7 = \frac{c_3}{6\cdot 7} = 0 \qquad \because c_3 = 0$

$k=6$, $\quad c_8 = \frac{c_4}{7\cdot 8} = \frac{1}{7\cdot 8}\cdot\frac{c_0}{3\cdot 4}$

$$= \frac{1}{3\cdot 4\cdot 7\cdot 8} c_0$$

$k=7$, $\quad c_9 = \frac{c_5}{8\cdot 9} = \frac{c_1}{4\cdot 5\cdot 8\cdot 9}$

$\vdots$

It can be show that

$$c_{4m} = \frac{1}{3\cdot 4\cdot 7\cdot 8\cdots(4m-1)4m} c_0 \qquad (m=1,2\ldots)$$

$$c_{4m+1} = \frac{1}{4\cdot 5\cdot 8\cdot 9\cdots 4m(4m+1)} c_1 \qquad (m=1,2\ldots)$$

$$\therefore \varphi(x) = c_0\left[1+\sum_{m=1}^{\infty}\frac{x^{4m}}{3\cdot 4\cdot 7\cdot 8\cdots(4m-1)4m}\right] + c_1 x + c_1\sum_{m=1}^{\infty}\frac{x^{4m+1}}{4\cdot 5\cdot 8\cdot 9\cdots 4m(4m+1)}$$

If we take $c_0=1$, $c_1=1$ we get,

$$\varphi(x) = 1+\sum_{m=1}^{\infty}\left[\frac{x^{4m}}{3\cdot 4\cdot 7\cdot 8\cdots(4m-1)4m}\right] + x + \sum_{m=1}^{\infty}\left[\frac{x^{4m+1}}{4\cdot 5\cdot 8\cdot 9\cdots 4m(4m+1)}\right]$$

These two solns are clearly linearly independent.

for if we define φ_1, φ_2 such that,

$$\varphi_1(0)=1, \quad \varphi_2(0)=0, \quad \varphi_1'(0)=0, \quad \varphi_2'(0)=1$$

$$\therefore W(\varphi_1,\varphi_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0.$$

1.b) Find two linearly independent power series solns about $x=0$ of the eqn $y''+xy'+x^2y=0$.

Soln:

Given $y''+xy'+x^2y=0 \rightarrow (1)$

Here $a_1(x)=1$, $a_2(x)=x^2$ both are analytic at $x=0$

$y''+y=0$. U_2 6m

Let $y=\phi(x)=\sum_{k=0}^{\infty} c_k x^k \rightarrow (2)$

$$y'=\sum_{k=0}^{\infty} k\cdot c_k x^{k-1} = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y''=\sum_{k=0}^{\infty} k(k-1) c_k x^{k-2} = \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2}$$

Sub in (1) we get,

$$\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} + x\sum_{k=1}^{\infty} k c_k x^{k-1} + x^2\sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=2}^{\infty} c_k k(k-1)x^{k-2} + \sum_{k=1}^{\infty} c_k x^k + \sum_{k=0}^{\infty} c_k x^{k+2} = 0$$

Replacing k by $k+2$ in first summation and k by $k-2$ in the third summation.

We get,

ie) $\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=2}^{\infty} c_{k-2}x^k = 0$

$$2c_2+(6c_3+c_1)x+\sum_{k=2}^{\infty}[(k+2)(k+1)c_{k+2}+kc_k+c_{k-2}]x^k=0 \rightarrow (3)$$

equating the coefficients of various powers to zero

we get, $c_2=0$, $c_3=-\frac{1}{6}c_1$

$(k+2)(k+1)c_{k+2}+kc_k+c_{k-2}=0$ for $k\geq 2$.

$$c_{k+2} = -\frac{kc_k+c_{k-2}}{(k+1)(k+2)}, \quad k=2,3,4\ldots$$

for $k=2$, $c_4 = -\frac{2c_2+c_0}{3\cdot 4} = -\frac{c_0}{12}$ $(\because c_2=0)$

$k=3$, $c_5 = \frac{-3c_3+c_1}{4\cdot 5} = \frac{-\frac{1}{2}c_1+c_1}{20} = \frac{-c_1}{40}$

$K=4$. $c_6 = \frac{-4c_4 + c_2}{5\cdot 6} = \frac{\frac{1}{3}c_0}{30} = \frac{c_0}{90}$ etc.

sub in ②

$$y = c_0 + c_1 x - \frac{1}{6}c_1 x^3 - \frac{1}{12}c_0 x^4 - \frac{1}{40}c_1 x^5 + \frac{c_0}{90}x^6 + \cdots$$

$$= c_0\left[1 - \frac{1}{12}x^4 + \frac{x^6}{90}\cdots\right] + c_1\left[x - \frac{x^3}{6} - \frac{1}{40}x^5 \cdots\right]$$

Where c_0 and c_1 are arbitrary constants

we may take $c_0 = c_1 = 1$

$$y = \left[1 - \frac{x^4}{12} + \frac{x^6}{90}\cdots\right] + \left[x - \frac{x^3}{6} - \frac{x^5}{40}\cdots\right]$$

1.c) Find The power series soln of the $(x^2+1)y'' + xy' - xy = 0$ in powers of x [(ie) about $x=0$]

Soln:

Given $(x^2+1)y'' + xy' - xy = 0 \rightarrow ①$

$$y'' + \frac{x}{1+x^2}y' - \frac{x}{1+x^2}y = 0$$

Here $a_1(x) = \frac{x}{1+x^2}$, $a_2(x) = \frac{-x}{1+x^2}$

Both are analytic at $x=0$ at since $\frac{1}{1+x^2} = (1+x^2)^{-1} = 1 - x^2 + x^4 \cdots$ which converges for $|x| < 1$

Let $y = \phi(x) = \sum_{k=0}^{\infty} c_k x^k \rightarrow ②$

$$y' = \phi'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \phi''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

sub in ①

$$(x^2+1)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + x\sum_{k=1}^{\infty} k c_k x^{k-1} - x\sum_{k=0}^{\infty} c_k x^k = 0$$

(ie) $\sum_{k=2}^{\infty} k(k-1)c_k x^k + \sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} + \sum_{k=1}^{\infty} k c_k x^k - \sum_{k=0}^{\infty} c_k x^{k+1} = 0$

replace k by $k+2$ in second summation and k by $k-1$ in the last summation.

i.e) $\sum_{k=2}^{\infty} k(k-1)C_k x^k + \sum_{k=0}^{\infty}(k+2)(k+1)C_{k+2}x^k + \sum_{k=1}^{\infty} kC_k x^k - \sum_{k=1}^{\infty} C_{k-1}x^k = 0$

$$2C_2 + (6C_3 + C_1 - C_0)x + \sum_{k=2}^{\infty}[k(k-1)C_k + (k+2)(k+1)C_{k+2} + kC_k - C_{k-1}]x^k = 0 \rightarrow (3)$$

This is true only if all the coefficients of the powers of x are zero.

$\therefore C_2 = 0$

$C_3 = \dfrac{C_0 - C_1}{6}$

and $C_{k+2} = \dfrac{(C_{k-1} - k^2 C_k)}{(k+1)(k+2)}$ for $k = 2, 3, 4 \ldots$ $\rightarrow (4)$

for $k=2$, $C_4 = \frac{1}{12}C_1$ $\because C_2 = 0$

$k=3$, $C_5 = \dfrac{-9C_3}{4\cdot 5} = \dfrac{-9}{4\cdot 5}\dfrac{(C_0 - C_1)}{6} = -\dfrac{3}{40}(C_0 - C_1)$ etc

Sub in (2)

$$y = C_0 + C_1 x + \frac{1}{6}(C_0 + C_1)x^3 + \frac{1}{12}C_1 x^4 - \frac{3}{40}(C_0 - C_1)x^5 + \ldots$$

$$= C_0\left[1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + \ldots\right] + C_1\left[x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3}{40}x^5 + \ldots\right]$$

Taking $C_0 = C_1 = 1$ we get,

$$y = \left[1 + \frac{x^3}{6} - \frac{3}{40}x^5 + \ldots\right] + \left[x - \frac{x^3}{6} + \frac{x^4}{12} + \frac{3}{40}x^5 + \ldots\right]$$

1.d) Find the soln φ of $xy'' + y' + 2y = 0$ in the form $\varphi(x) = \sum_{k=0}^{\infty} C_k(x-1)^k$, satisfying $\varphi(1) = 1$, $\varphi'(1) = 2$.

Soln:

$xy'' + y' + 2y = 0 \rightarrow (1)$

$y'' + \frac{1}{x}y' + \frac{2}{x}y = 0$

Here $a_1(x) = \frac{1}{x}$, $a_2(x) = \frac{2}{x}$ both are analytic at $x=1$

Since they can be expanded in powers of $(x-1)$

Let $y=\varphi(x)=\sum_{k=0}^{\infty} c_k(x-1)^k \rightarrow (2)$

$y'=\sum_{k=1}^{\infty} k c_k (x-1)^{k-1} \rightarrow (3)$

$y''=\sum_{k=2}^{\infty} k(k-1) c_k (x-1)^{k-2} \rightarrow (4)$

Sub in (1)

$$x\sum_{k=2}^{\infty} k(k-1)c_k(x-1)^{k-2}+\sum_{k=1}^{\infty} k c_k(x-1)^{k-1}+2\sum_{k=0}^{\infty} c_k(x-1)^k=0$$

$$[(x-1)+1]\sum_{k=2}^{\infty} k(k-1)c_k(x-1)^{k-2}+\sum_{k=1}^{\infty} k c_k(x-1)^{k-1}+\sum_{k=0}^{\infty} 2c_k(x-1)^k=0$$

ie)
$$\left.\begin{aligned}&\sum_{k=2}^{\infty} k(k-1)c_k(x-1)^{k-1}+\sum_{k=2}^{\infty} k(k-1)c_k(x-1)^{k-2}\\&+\sum_{k=1}^{\infty} k c_k(x-1)^{k-1}+\sum_{k=0}^{\infty} 2c_k(x-1)^k\end{aligned}\right\}=0$$

we replace k by k+1 in the first summation by k+2 in the second and by k+1 in the third summation.

ie)
$$\left.\begin{aligned}&\sum_{k=1}^{\infty}(k+1)k\,c_{k+1}(x-1)^k+\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}(x-1)^k\\&+\sum_{k=0}^{\infty}(k+1)c_{k+1}(x-1)^k+\sum_{k=0}^{\infty}2c_k(x-1)^k\end{aligned}\right\}=0$$

equating to zero the coefficients of various powers of $(x-1)$ we obtain.

$2c_2+c_1+2c_0=0$. $\therefore c_2=\dfrac{-(c_1+2c_0)}{2} \rightarrow (5)$

$(k+1)k\,c_{k+1}+(k+2)(k+1)c_{k+2}+(k+1)c_{k+1}+2c_k=0$ for $k\geq 1$

ie) $(k+1)(k+2)c_{k+2}+(k+1)^2c_{k+1}+2c_k=0$

$$c_{k+2}=\frac{-[(k+1)^2c_{k+1}+2c_k]}{(k+1)(k+2)} \rightarrow (6) \quad k=1,3,4\ldots$$

using $\varphi(1)=y(1)=1$, $\varphi'(1)=y'(1)=2$ in (2) and (3)

we get $c_0=1$

$\therefore c_1=2$

from ⑤ ⇒ $C_2 = \frac{-(2+2)}{2} = -2$

putting $k = 1, 2, 3 \ldots$ in ⑥

$$C_3 = \frac{-[4C_2+2C_1]}{2.3} = \frac{-[4(-2)+(2\times2)]}{2.3} = \frac{2}{3}$$

$$C_4 = \frac{-[9C_3+2C_2]}{3.4} = \frac{-[9(2/3)+2(-2)]}{3.4} = -\frac{1}{6}$$

$$C_5 = \frac{-(16C_4+2C_3)}{4.5} = \frac{-[16(-1/6)+2(2/3)]}{4.5} = \frac{1}{5}$$

Sub in ②

$$y = \phi(x) = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{1}{4}(x-1)^5 + \ldots$$

1.e) chebyshev eqn:

The eqn $(1-x^2)y'' - xy' + \alpha^2 y = 0$, where α is a constant is called the chebyshev eqn.

(a) compute two linearly independent series solns for $|x|<1$

(b) S.T for every non-negative integer $\alpha = n$, there is a polynomial soln of degree n, when appropriately normalized These are called the chebyshev polynomials.

Soln:

Given $(1-x^2)y'' - xy' + \alpha^2 y = 0 \rightarrow$ ①

(i) $y'' - \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0$

Here $a_1(x) = \frac{-x}{1-x^2}$, $a_2(x) = \frac{\alpha^2}{1-x^2}$, both are analytic at $x=0$

Since they have power series expansion valid in $|x|<1$

let $y = \phi(x) = \sum_{k=0}^{\infty} C_k x^k \rightarrow$ ② $= C_0 + C_1x + C_2x^2 + \ldots + C_k x^k + \ldots$

$$y' = \sum_{k=1}^{\infty} k C_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) C_k x^{k-2}$$

from (5) $\Rightarrow$ $c_2 = \frac{-(2+2)}{2} = -2$

putting $k=1,2,3\ldots$ in (6)

$$c_3 = \frac{-[4c_2+2c_1]}{2.3} = \frac{-[4(-2)+(2\times 2)]}{2.3} = \frac{2}{3}$$

$$c_4 = \frac{-[9c_3+2c_2]}{3.4} = -\frac{[9(2/3)+2(-2)]}{3.4} = -\frac{1}{6}$$

$$c_5 = \frac{-(16c_4+2c_3)}{4.5} = -\frac{[16(-1/6)+2(2/3)]}{4.5} = \frac{1}{5}$$

Sub in (2)

$$y = \varphi(x) = 1+2(x-1)-2(x-1)^2+\frac{2}{3}(x-1)^3-\frac{1}{6}(x-1)^4+\frac{1}{4}(x-1)^5+\ldots$$

1.e) Chebyshev eqn:

The eqn $(1-x^2)y''-xy'+\alpha^2 y=0$, where α is a constant is called the chebyshev eqn.

(a) Compute two linearly independent series solns for $|x|<1$

(b) S.T for every non-negative integer $\alpha=n$, there is a polynomial soln of degree n, when appropriately normalized. These are called the chebyshev polynomials.

Soln:

Given $(1-x^2)y''-xy'+\alpha^2 y=0 \rightarrow$ (1)

(i) $y'' - \frac{x}{1-x^2}y' + \frac{\alpha^2}{1-x^2}y = 0$

Here $a_1(x) = \frac{-x}{1-x^2}$, $a_2(x) = \frac{\alpha^2}{1-x^2}$, both are analytic at $x=0$

Since they have power series expansion valid in $|x|<1$

Let $y = \varphi(x) = \sum_{k=0}^{\infty} c_k x^k \rightarrow$ (2) $= c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k + \ldots$

$$y' = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$y'' = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

Sub in ①

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - x\sum_{k=1}^{\infty} k c_k x^{k-1} + \alpha^2 \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1)c_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} \alpha^2 c_k x^k = 0$$

$k = k+2$

replacing k by $k+2$ in the first summation.

i.e. $$\sum_{k=0}^{\infty}(k+2)(k+1)c_{k+2}x^k - \sum_{k=2}^{\infty} k(k-1)c_k x^k - \sum_{k=1}^{\infty} k c_k x^k + \sum_{k=0}^{\infty} \alpha^2 c_k x^k = 0$$

$k=0$ $\qquad$ $k=1$

i.e. $$(2c_2+\alpha^2 c_0) + (3.2c_3+\alpha^2 c_1 - c_1)x + \sum_{k=2}^{\infty}[(k+1)(k+2)c_{k+2} - k^2 c_k + \alpha^2 c_k]x^k = 0$$

This is true only if all the coefficients of powers of x are zero

$$\therefore\ c_2 = -\frac{\alpha^2}{2}c_0, \quad c_3 = \frac{(1-\alpha^2)c_1}{6} = (1-\alpha^2)\frac{c_1}{\lfloor 3}$$

$$c_{k+2} = \frac{(k^2-\alpha^2)c_k}{(k+1)(k+2)} \quad \text{for } k=2,3,4\ldots$$

$2\,c_0$ $\quad = -\alpha^2/2$

for $k=2$, $$c_4 = \frac{(2^2-\alpha^2)c_2}{3\cdot 4} = \frac{(-\alpha^2)(2^2-\alpha^2)}{\lfloor 4}c_0$$ $\quad$ [c_2

for $k=3$, $$c_5 = \frac{(3^2-\alpha^2)c_3}{4\cdot 5} = \frac{(1-\alpha^2)(3^2-\alpha^2)}{\lfloor 3\cdot 4\cdot 5}c_1 = \frac{(1-\alpha^2)(3^2-\alpha^2)c_1}{\lfloor 5}$$

In general, $$c_{2m} = \frac{(-\alpha^2)(2^2-\alpha^2)\cdots[(2m-2)^2-\alpha^2]c_0}{\lfloor 2m}, \quad m=1,2,3$$

c_2, c_4, c_6

$$c_{2m+1} = \frac{(1^2-\alpha^2)(3^2-\alpha^2)\cdots[(2m-1)^2-\alpha^2]c_1}{\lfloor 2m+1}, \quad m=1,2$$

c_3, c_5, c_7

Taking $c_0=c_1=1$ $\quad y=\phi(x) = c_0 + c_1 x + c_2 x^2 + \cdots + f(x)x^k + \cdots$

$= 1 + x + \sum c_{2m} + \sum c_{2m+1}$

sub in ② we get,

$$y = \phi(x) = 1 + \sum_{m=1}^{\infty}\frac{(-\alpha^2)(2^2-\alpha^2)\cdots[(2m-2)^2-\alpha^2]}{\lfloor 2m}x^{2m} + x$$

$$+ \sum_{m=1}^{\infty}\frac{(1^2-\alpha^2)(3^2-\alpha^2)\cdots[(2m-1)^2-\alpha^2]}{\lfloor 2m+1}x^{2m+1}$$

Sub in ①

$$(1-x^2)\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} - x\sum_{k=1}^{\infty} kC_k x^{k-1} + \alpha^2\sum_{k=0}^{\infty} C_k x^k = 0$$

$$\sum_{k=2}^{\infty} k(k-1)C_k x^{k-2} - \sum_{k=2}^{\infty} k(k-1)C_k x^k - \sum_{k=1}^{\infty} kC_k x^k + \sum_{k=0}^{\infty} \alpha^2 C_k x^k = 0$$

$\downarrow$

$k=k+2$

replacing k by $k+2$ in the first summation.

(i) $$\sum_{k=0}^{\infty} (k+2)(k+1)C_{k+2} x^k - \sum_{k=2}^{\infty} k(k-1)C_k x^k - \sum_{k=1}^{\infty} kC_k x^k + \sum_{k=0}^{\infty} \alpha^2 C_k x^k = 0$$

$k=0$ $\quad$ $k=1$

(ii) $$\left(2C_2+\alpha^2 C_0\right) + \left(3.2C_3+\alpha^2 C_1 - C_1\right)x + \sum_{k=2}^{\infty}\left[(k+1)(k+2)C_{k+2} - k^2C_k + \alpha^2 C_k\right]x^k = 0$$

This is true only if all the coefficients of powers of x are zero

$\therefore\ C_2 = -\frac{\alpha^2}{2}C_0, \quad C_3 = \frac{(1-\alpha^2)C_1}{6} = (1-\alpha^2)\frac{C_1}{\lfloor 3}$

$$C_{k+2} = \frac{(k^2-\alpha^2)C_k}{(k+1)(k+2)} \quad \text{for } k=2,3,4\ldots$$

for $k=2$, $C_4 = \frac{(2^2-\alpha^2)C_2}{3\cdot 4} = \frac{(-\alpha^2)(2^2-\alpha^2)}{\lfloor 4}C_0$

for $k=3$, $C_5 = \frac{(3^2-\alpha^2)C_3}{4\cdot 5} = \frac{(1-\alpha^2)(3^2-\alpha^2)}{\lfloor 3\cdot 4\cdot 5}C_1 = \frac{(1-\alpha^2)(3^2-\alpha^2)C_1}{\lfloor 5}$

In general, $C_{2m} = \frac{(-\alpha^2)(2^2-\alpha^2)\cdots\left[(2m-2)^2-\alpha^2\right]C_0}{\lfloor 2m}$, $m=1,2,3$

C_2, C_4, C_6

$C_{2m+1} = \frac{(1^2-\alpha^2)(3^2-\alpha^2)\cdots\left[(2m-1)^2-\alpha^2\right]C_1}{\lfloor 2m+1}$, $m=1,2$

C_3, C_5, C_7

$y=\phi(x) = C_0+C_1x+C_2x^2+\cdots+C_kx^k$

$= 1+x+\sum C_{2m}x^{2m}+\sum C_{2m+1}x^{2m+1}$

Taking $C_0=C_1=1$

sub in ① we get,

$$y=\phi(x) = 1+\sum_{m=1}^{\infty}\frac{(-\alpha^2)(2^2-\alpha^2)\cdots\left[(2m-2)^2-\alpha^2\right]}{\lfloor 2m}x^{2m} + x$$

$$+\sum_{m=1}^{\infty}\frac{(1^2-\alpha^2)(3^2-\alpha^2)\cdots\left[(2m-1)^2-\alpha^2\right]}{\lfloor 2m+1}x^{2m+1}$$

). If $\alpha = n$ an even integer then

$$\varphi_1(x) = 1 + \sum \frac{(-\alpha^2)(2^2-\alpha^2)\cdots[(2m-2)^2-\alpha^2]}{\lfloor 2m} x^{2m}$$ is a polynomial

If $\alpha = n$ is an odd integer

$$\varphi_2(x) = x + \sum \frac{(1-\alpha^2)(3^2-\alpha^2)\cdots[(2m-1)^2-\alpha^2]}{\lfloor 2m+1} x^{2m+1}$$

is a Polynomial

f) Hermite eqn:

The eqn $y'' - 2xy' + 2\alpha y = 0$, where α is a constant is called the hermite eqn

(a) Find two linearly independent soln on $-\infty < x < \infty$

(b) S.T There is a polynomial soln of degree n incase $\alpha = n$ is a non-negative integer.

(c) S.T The polynomial H_n defined by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n}(e^{-x^2})$ is a soln of Hermite eqn in case $\alpha = n$ is non-negative integer.

(d) compute H_0, H_1, H_2, H_3

Soln:

a) Given $y'' - 2xy' + 2\alpha y = 0 \rightarrow (1)$

Let $y = \varphi(x) = \sum_{m=0}^{\infty} C_m x^{k+m}$, $C_0 \neq 0 \rightarrow (2)$

Diff (2) and sub in (1) weget,

$$\left.\begin{aligned} &\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} - 2x\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} \\ &+ 2\alpha \sum_{m=0}^{\infty} C_m x^{k+m} \end{aligned}\right\} = 0$$

(ie) $$\left.\begin{aligned} &\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} - 2\Big[\sum_{m=0}^{\infty} C_m(k+m)x^{k+m} \\ &- \sum_{m=0}^{\infty} C_m \alpha x^{k+m}\Big] \end{aligned}\right\} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} c_m(k+m)(k+m-1)x^{k+m-2} - 2\sum_{m=0}^{\infty} c_m(k+m-\alpha)x^{k+m} = 0 \rightarrow (3)$$

The indicial eqn is obtained by eqn to zero the coefficient of lowest power of x.

Equating to zero next smallest power of x term namely coefficient of x^{k-1} in (3) we obtain

$$k(k-1)c_0 = 0 \quad (\text{or}) \quad k(k-1) = 0 \quad \text{as } c_0 \neq 0 \rightarrow (4)$$

The roots of indicial eqn (4) are $k=0,1$ They are distinct and differ by an integer. equating to zero next smallest power of x term namely coefficient of x^{k-1} in (3) we obtain,

$$c_1(k-1)k = 0 \rightarrow (5)$$

When $k=0$ in (5)

Shows that c_1 is indeterminate

Hence c_0 and c_1 may be taken as arbitrary constants. equating to zero the coefficients of x^{k+m-2} in (3)

We have,

$$c_m(k+m)(k+m-1) - 2c_{m-2}(k+m-2-\alpha) = 0$$

$$c_m = \frac{2(k+m-2-\alpha)}{(k+m)(k+m-1)} c_{m-2} \rightarrow (6)$$

Put $k=0$ in (6)

$$c_m = \frac{2(m-2-\alpha)}{m(m-1)} c_{m-2} \rightarrow (7)$$

Putting $m=2,4,6 \ldots$ in (7)

We have,

$$c_2 = -\frac{2\alpha}{\underline{|2}} c_0 = \frac{(-1)^1 2^1 \alpha c_0}{\underline{|2}}$$

$$c_4 = \frac{2(2-\alpha)}{4\cdot 3} c_2 = \frac{2(2-\alpha)2\alpha c_0}{3\cdot 4 \underline{|2}}$$

$$c_4 = \frac{(-1)^2 2^2 \alpha(\alpha-2)}{\underline{|4}} c_0$$

$\vdots$

$$C_{2m} = \frac{(-1)^m 2^m \alpha(\alpha-2)\cdots(\alpha-2m+2)}{\lfloor 2m} C_0$$

Next putting $n = 3, 5, 7 \ldots (2m+1)$ in ⑦

$$C_3 = \frac{2(1-\alpha)}{3\cdot 2} C_1 = \frac{(-1)^1 2^1 (\alpha-1)}{\lfloor 3} C_1$$

$$C_5 = \frac{2(3-\alpha)}{5\cdot 4} C_3 = \frac{(-1)^2 2^2 (\alpha-1)(\alpha-3)}{\lfloor 5} C_1$$

Putting $k=0$ in ③ and sub the above values

we get,

$$y = \varphi(x) = (C_0 + C_2x^2 + C_4x^4 + \ldots) + (C_1x + C_3x^3 + C_5x^5 + \ldots)$$

$$= C_0\left[1 - \frac{2\alpha}{\lfloor 2}x^2 + \frac{2^2\alpha(\alpha-2)}{\lfloor 4}x^4 + \ldots + \frac{(-2)^m\alpha(\alpha-2)\ldots(\alpha-2m+2)}{\lfloor 2m}x^{2m} + \ldots\right]$$

$$+ C_1\left[x - \frac{2(\alpha-1)}{\lfloor 3}x^3 + \frac{2^2(\alpha-1)(\alpha-3)}{\lfloor 5}x^5 + \ldots + \frac{(-2)^m(\alpha-1)(\alpha-3)\ldots(\alpha-2m+1)}{\lfloor 2m+1}x^{2m+1} + \ldots\right]$$

$\rightarrow$ ⑧

(b) Replacing n by $m+2$ in ⑦

we have,

$$C_{m+2} = \frac{2(m-\alpha)}{(m+1)(m+2)} C_m \rightarrow ⑨$$

Let $\alpha = n$ be non negative. Then ⑨ shows that C_{m+2} and all subsequent coefficients in ⑦ will vanish and the corresponding series terminate.

We shall obtain the series soln of ① in descending powers of x by assuming $\alpha = n$ be a non-negative integer. from ③ with $k=0$

we have,

$$y = \varphi(x) = C_n x^n + C_{n-2}x^{n-2} + C_{n-4}x^{n-4} + \ldots \rightarrow ⑩$$

from (9)

$$c_m = -\frac{(m+1)(m+2)}{2(n-m)} c_{m+2} \rightarrow (11)$$

Putting $m = n-2, n-4 \dots$ in (11)

We have

$$c_{n-2} = \frac{-n(n-1)}{2(n-n+2)} c_n = \frac{n(n-1)}{2.2} c_n$$

$$c_{n-4} = \frac{-(n-3)(n-2)}{2.(n-n+4)} c_{n-2} = \frac{-n(n-1)(n-2)(n-3)}{2^2.2.4} c_n \text{ and so on}$$

Sub in (10)

We obtain,

$$\varphi(x) = c_n \Big\{ x^n - \frac{n(n-1)}{2.2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2.2.4} x^{n-4} + \dots$$

$$+ (-1)^r \frac{n(n-1)(n-2r+1)}{2^r.2.4 \dots 2r} x^{n-2r} + \dots \Big\}$$

$$= c_n \sum_{r=0}^{[\frac{n}{2}]} (-1)^r . \frac{n(n-1) \dots (n-2r+1)}{2^r.2.4 \dots 2r} x^{n-2r}$$

$$= c_n \sum_{r=0}^{[\frac{n}{2}]} \frac{(-1)^r \lfloor n}{2^{2r} \lfloor r \lfloor n-2r} x^{n-2r}$$

as $[\frac{n}{2}] = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$

Taking $a_n = 2^n$ and denoting the Soln by $H_n(x)$. We obtain the standard soln of (1) known as the n^{th} Hermite Polynomial of order n.

$$H_n(x) = \sum_{r=0}^{[\frac{n}{2}]} (-1)^n \frac{\lfloor n}{\lfloor n-2r} (2x)^{n-2r}$$

(c) $H_n = (-1)^n e^{x^2} D^n (e^{-x^2})$ where $D = \frac{d}{dx}$

$$H_n' = (-1)^n [e^{x^2} D^{n+1} (e^{-x^2}) + D^n (e^{-x^2}) e^{x^2} \cdot 2x]$$

$$H_n' = -H_{n+1} + 2x H_n$$

$$H_n'' = (-1)^n [e^{x^2} D^{n+2} (e^{-x^2}) + 4x e^{x^2} D^{n+1} (e^{-x^2}) + D^n (e^{-x^2}) \{2e^{2x} + 4x e^{x^2}\}]$$

$$H_n'' = H_{n+2} - 4x H_{n+1} + 2H_n + 4x H_n$$

$$\therefore H_n'' - 2x H_n' + 2n H_n = H_{n+2} - 4x H_{n+1} + 2H_n + 4x H_n - 2x(-H_{n+1} + 2x H_n) + 2n H_n$$

$$H_n'' - 2x H_n' + 2n H_n = H_{n+2} - 2x H_{n+1} + (n+1) H_n \rightarrow (*)$$

Now let,

$$u(x) = e^{-x^2}$$

$$u'(x) = -2x e^{-x^2} = -2x u(x)$$

$$\therefore u'(x) + 2x u(x) = 0$$

Diff this n times using leibnitz theorem,

weget,

$$u_{n+1}(x) + 2x u_n(x) + 2n u_{n-1} = 0 \rightarrow (12)$$

Wehave $H_n(x) = (-1)^n e^{x^2} D^n (e^{-x^2})$

$$= (-1)^n e^{x^2} u_n$$

$$u_n = (-1)^n e^{-x^2} H_n(x)$$

Sub in (12) weget,

$$(-1)^{n+1} e^{-x^2} H_{n+1}(x) + 2x (-1)^n e^{-x^2} H_n(x) + 2n (-1)^{n-1} e^{-x^2} H_{n-1}(x) = 0$$

i) $H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$

Thus we have to recurrence relation.

$$H_{n+1}(x) - 2x H_n(x) + 2n H_{n-1}(x) = 0$$

changing n to n+1

$$H_{n+2}(x) - 2x H_{n+1}(x) + 2(n+2) H_n(x) = 0$$

$\therefore$ (*) becomes $H_n'' - 2xH_n' + 2nH_n = 0$

Thus $H_n(x)$ satisfies the Hermite eqn

$y'' - 2xy' + 2ny = 0$

(d) Putting $n = 0, 1, 2$ and 3 in $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} (e^{-x^2})$

we have.

$H_0 = e^{x^2} e^{-x^2} = 1$

$H_1 = (-1) e^{x^2} \frac{d}{dx} (e^{-x^2}) = 2x$

$H_2 = (-1)^2 e^{x^2} \frac{d^2}{dx^2} (e^{-x^2})$

$= e^{x^2} \frac{d}{dx} (-2xe^{-x^2}) = e^{x^2} \{4x^2 e^{-x^2} - 2xe^{-x^2}\}$

$H_2 = \{4x^2 - 2\}$

$H_3 = (-1)^3 e^{x^2} \frac{d^3}{dx^3} (e^{-x^2}) = -e^{x^2} \frac{d}{dx} \left(\frac{d^2}{dx^2} (e^{-x^2})\right)$

$= -e^{x^2} \frac{d}{dx} (4x^2 e^{-x^2} - 2xe^{-x^2})$

$= -e^{x^2} \frac{d}{dx} \{(4x^2 - 2) e^{-x^2}\}$

$= -e^{x^2} \{-2x (4x^2 - 2) e^{-x^2} + 8xe^{-x^2}\}$

$H_3 = 8x^3 - 12x$

Section-8

The legendre equation.

The eqn $L(y) = (1 - x^2) y'' - 2xy' + \alpha(\alpha + 1) y = 0 \rightarrow ①$

where α is constant is called the legendre equation.

writing the eqn as

$$y'' - \frac{2x}{1-x^2} y' + \frac{\alpha(\alpha+1)}{1-x^2} y = 0$$

Here $a_1(x) = \frac{-2x}{1-x^2}$, $a_2(x) = \frac{\alpha(\alpha+1)}{1-x^2}$

Both are analytic at $x = 0$

since $\frac{1}{1-x^2} = (1-x^2)^{-1} = 1 + x^2 + x^4 + \cdots = \sum_{k=0}^{\infty} x^{2k}$

and this series converges for $|x|<1$

$$\therefore \; a_1(x) = \sum_{k=0}^{\infty} (-2)\, x^{2k+1} \quad , \quad a_2(x) = \sum_{k=0}^{\infty} \alpha(\alpha+1) x^{2k}$$

which converge for $|x|<1$

By thm : 12 it follows that the solns of $L(y)=0$ on $|x|<1$ have convergent power series expansions

Let $\varphi(x) = \sum_{k=0}^{\infty} c_k x^k \rightarrow (2)$ be any soln of $L(y)=0$, on $|x|<1$

$$\varphi'(x) = \sum_{k=1}^{\infty} k c_k x^{k-1}$$

$$\varphi''(x) = \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2}$$

Sub in (1) we get,

$$L(\varphi(x)) = (1-x^2)\varphi''(x) - 2x\varphi'(x) + \alpha(\alpha+1)\varphi(x)$$

$$= (1-x^2) \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - 2x \sum_{k=1}^{\infty} k c_k x^{k-1} + \alpha(\alpha+1) \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=2}^{\infty} k(k-1) c_k x^{k-2} - \sum_{k=0}^{\infty} k(k-1) c_k x^k - 2 \sum_{k=0}^{\infty} k c_k x^k + \alpha(\alpha+1) \sum_{k=0}^{\infty} c_k x^k$$

$$= \sum_{k=0}^{\infty} \left[(k+2)(k+1) c_{k+2} - k(k-1) c_k - 2k c_k + \alpha(\alpha+1) c_k \right] x^k$$

for φ to satisfy $L(y)=0$, we must have all the coefficients of the powers of x equal to zero.

Hence

$$(k+2)(k+1) c_{k+2} + (\alpha+k+1)(\alpha-k) c_k = 0 \rightarrow (3) \quad k = 0, 1, 2 \ldots$$

$$c_{k+2} = -\frac{(\alpha+k+1)(\alpha-k)}{(k+1)(k+2)} c_k$$

Put $k=0$, $c_2 = -\frac{(\alpha+1)\alpha}{2} c_0$

for, $k=1$, $c_3 = \frac{-(\alpha+2)(\alpha-1)}{3\cdot 2} c_1$

$k=2$, $c_4 = \frac{-(\alpha+3)(\alpha-2)}{4\cdot 3} c_2$

$= \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{4\cdot 3\cdot 2} c_0$

$k=3$, $c_5 = \frac{-(\alpha+4)(\alpha-3)}{5\cdot 4} c_3$

$= \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{5\cdot 4\cdot 3\cdot 2} c_1$ etc

It follows by induction that for $m=1,2,3$.

$$c_{2m} = \frac{(-1)^m(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{\lfloor 2m} c_0$$

and

$$c_{2m+1} = \frac{(-1)^m(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{\lfloor 2m+1} c_1$$

we have determined all the coefficients in terms of c_0 and c_1

$\therefore \quad \varphi(x) = c_0\varphi_1(x) + c_1\varphi_2(x)$

Where

$$\varphi_1(x) = 1 + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m-1)(\alpha+2m-3)\cdots(\alpha+1)\alpha(\alpha-2)\cdots(\alpha-2m+2)}{\lfloor 2m} x^{2m}$$

$$\Rightarrow \varphi_1(x) = 1 - \frac{(\alpha+1)}{\lfloor 2} x^2 + \frac{(\alpha+3)(\alpha+1)\alpha(\alpha-2)}{\lfloor 4} x^4 \cdots$$

and

$$\varphi_2(x) = x + \sum_{m=1}^{\infty} (-1)^m \frac{(\alpha+2m)(\alpha+2m-2)\cdots(\alpha+2)(\alpha-1)(\alpha-3)\cdots(\alpha-2m+1)}{\lfloor 2m+1} x^{2m+1}$$

$$\Rightarrow \varphi_2(x) = x - \frac{(\alpha+2)(\alpha-1)}{\lfloor 3} x^3 + \frac{(\alpha+4)(\alpha+2)(\alpha-1)(\alpha-3)}{\lfloor 5} x^5 + \cdots$$

Both φ_1 and φ_2 are solns of the legendre eqn.

These corresponding to the choices $C_0=1, C_1=0$ and $C_0=0, C_1=1$ respectively.

They form a basis

since $\varphi_1(0)=1$. $\varphi_2(0)=0$

$\varphi_1'(0)=0$, $\varphi_2'(0)=1$

and $W(\varphi_1,\varphi_2)(0)=1\neq 0$

Result:

The Polynomial soln P_n of degree n of $(1-x^2)y''-2xy'+n(n+1)y=0$ satisfying $P_n(1)=1$ is called the n^{th} legendre Polynomial.

Legendre Polynomial of n^{th} degree is denoted and defined by

$$P_n(x)=\frac{1\cdot 3\cdot 5\cdots(2n-1)}{\lfloor n}\left[x^n-\frac{n(n-1)}{2\cdot(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}-\cdots\right]$$

The above polynomial can be put in the form

$$P_n(x)=\sum_{r=0}^{[\frac{n}{2}]}\frac{(-1)^r\lfloor 2n-2r}{2^n\lfloor r\lfloor n-r\lfloor n-2r}x^{n-2r}$$

where $[\frac{n}{2}]=\frac{n}{2}$ if n is even

$\frac{n-1}{2}$ if n is odd

Determination of first few legendre polynomials.

We have $P_n(x)=\dfrac{1\cdot 3\cdot 5\cdots(2n-1)}{\lfloor n}\left[x^n-\dfrac{n(n-1)}{2(2n-1)}x^{n-2}+\dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4}\cdots\right]$ $\to (1)$

Putting $n=0,1,2\ldots$ in (1) we obtain.

$$P_0(x)=\frac{1}{\lfloor 0}x^0=1$$

$$P_1(x)=\frac{1}{\lfloor 1}x^1=x$$

$$P_2(x) = \frac{1\cdot 3}{\lfloor 2}\left[x^2 - \frac{2\cdot 1}{2\cdot 3}x^0\right] = \frac{1}{2}(3x^2-1)$$

$$P_3(x) = \frac{1\cdot 3\cdot 5}{\lfloor 3}\left[x^3 - \frac{3\cdot 2}{2\cdot 5}x^1\right] = \frac{1}{2}[5x^3-3x]$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{\lfloor 4}\left[x^4 - \frac{4\cdot 3}{2\cdot 7}x^2 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}x^0\right]$$

$$= \frac{1}{8}(35x^4 - 30x^2 + 3)$$

$$P_5(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{\lfloor 5}\left[x^5 - \frac{5\cdot 4}{2\cdot 9}x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 9\cdot 7}x^1\right]$$

$= \frac{1}{8}(63x^5 - 70x^3 + 15x)$ and so on.

Problems.

1.a) Express $2-3x+4x^2$ in terms of legendre Polynomial

Soln:

We have $P_0(x)=1$, $P_1(x)=x$, $P_2(x)=\frac{1}{2}(3x^2-1)$

Put $2-3x+4x^2 = A\cdot P_0(x) + B\cdot P_1(x) + C\cdot P_2(x)$

$2-3x+4x^2 = A\cdot 1 + Bx + \frac{C}{2}(3x^2-1)$

equating like powers on both sides

$2 = A - \frac{C}{2}$

$2A - C = 4 \rightarrow (1)$

$B = -3$

$4 = C(3/2) \Rightarrow C = 8/3$

Sub in (1)

$2A - \frac{8}{3} = 4$

$2A = 4 + \frac{8}{3}$

$2A = \frac{20}{3}$

$A = \frac{10}{3}$

$\therefore 2-3x+4x^2 = \frac{10}{3} + P_0(x) - 3P_1(x) + \frac{8}{3}P_2(x)$

(b) Rodrigue's formula

S.T $P_n(x) = \frac{1}{2^n \underline{n}} \frac{d^n}{dx^n}(x^2-1)^n$

$D^n(uv) = u^n v + nC_1 u^{n-1} v_1 + \cdots$

Soln:

Let $u(x) = (x^2-1)^n \rightarrow ①$

Diff w.r.t x

$u' = n(x^2-1)^{n-1} 2x$

$(a+b)^n = a^n + nC_1 a^{n-1} b + nC_2 a^{n-2} b^2 + \cdots$

$(x^2-1)u' = 2nx(x^2-1)^n$

$= 2nx.u$

$(x^2-1)u' - 2nxu = 0$

$\Rightarrow (1-x^2)u' + 2nxu = 0$

Diff this $(n+1)$ times $[(1-x^2)u']^{n+1} + (2nxu)^{n+1} = 0$

$$\left.\begin{aligned}(1-x^2)u^{(n+2)} + (n+1)(-2x)u^{(n+1)} + \frac{(n+1)n}{1.2}(-2)u^{(n)} \\ + 2nxu^{(n+1)} + 2n.(n+1)u^{(n)}\end{aligned}\right\} = 0$$

$V = 1-x^2$

$u = u'$

$(1-x^2)u^{(n+2)} + 2x\boxed{\cancel{-1}-1+n}u^{(n+1)} + u^{(n)}[n(n+1)] = 0$

i) $(1-x^2)u^{(n+2)} - 2xu^{(n+1)} + n(n+1)u^{(n)} = 0$

If $\varphi = u^{(n)}$

we obtain,

$$(1-x^2)\varphi''(x) - 2x\varphi'(x) + n(n+1)\varphi(x) = 0$$

$\therefore \varphi(x) = \frac{d^n}{dx^n}(x^2-1)^n$ satisfies the legendre eqn

$$(1-x^2)y'' - 2xy'(x) + n(n+1)y = 0 \rightarrow ②$$

But we know that for each integer n.

$P_n(x)$ is the only Polynomial soln for the legendre eqn

$P_n(x) = K.\varphi(x)$

$P_n(x) = K.\frac{d^n}{dx^n}(x^2-1)^n$

To determine K

$P_n(x) = K[D^n(x-1)^n(x+1)^n]$ where $D = \frac{d}{dx}$

$= k[(x+1)^n D^n (x-1)^n +$ terms with $(x-1)$ as a factor$]$

$P_n(x) = k[(x+1)^n \lfloor n +$ terms with $(x-1)$ as a factor$]$

Putting $x=1$,

$P_n(1) = k\, 2^n \lfloor n$, But $P_n(1) = 1$

$1 = k\, 2^n \lfloor n$

$k = \dfrac{1}{2^n \lfloor n}$

$\therefore P_n(x) = k\,\phi(x) = \dfrac{1}{2^n \lfloor n} \dfrac{d^n}{dx^n} (x^2-1)^n$

Note:

$(1-x^2)y'' - 2xy' + n(n+1)y = 0$ There is no other Polynomial which is at $x > 1$

1.c) Generating function for legendre Polynomials

S.T $P_n(x)$ is the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$ in ascending powers of z

(i) $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$, $|x| \le 1$, $|z| < 1$

Soln:

Since $|z| < 1$ and $|x| \le 1$

We have,

$(1-2xz+z^2)^{-1/2} = [1 - z(2x-z)]^{-1/2}$

$= 1 + \frac{1}{2} z(2x-z) + \dfrac{1\cdot 3}{2\cdot 4} z^2 (2x-z)^2 + \dots + \dfrac{1\cdot 3 \dots (2n-3)}{2\cdot 4 \dots (2n-2)} z^{n-1}(2x-z)^{n-1}$

$+ \dfrac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots 2n} z^n (2x-z)^n + \dots$

coefficient of z^n in $\dfrac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots 2n} z^n (2x-z)^n$

$= \dfrac{1\cdot 3 \dots (2n-1)}{2\cdot 4 \dots 2n} (2x)^n$

$= \dfrac{1\cdot 3 \dots (2n-1)}{(2\cdot 1)(2\cdot 2)\dots(2\cdot n)} 2^n x^n$

$$= \frac{1.3.5\ldots(2n-1)}{\lfloor n} x^n \rightarrow ①$$

again coefficient of z^n in $\frac{1.3\ldots(2n-3)}{2.4\ldots(2n-2)} z^{n-1}(2x-z)^{n-1}$

$$= \frac{1.3\ldots(2n-3)}{(2.1)(2.2)\ldots2(n-1)}\left[-(n-1)(2x)^{n-2}\right]$$

$$= -\frac{1.3\ldots(2n-3)}{2^{n-1}\,1.2\ldots(n-1)}\left[\frac{2n-1}{n}\cdot\frac{n}{2n-1}\right](n-1)2^{n-2}x^{n-2}$$

$$= -\frac{1.3\ldots(2n-1)}{\lfloor n}\frac{n(n-1)}{2.(2n-1)}x^{n-2} \rightarrow ② \text{ and so on.}$$

using ①, ②, we find that the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$ is given by

$$(1-2xz+z^2)^{-1/2} = \frac{1.3\ldots(2n-1)}{\lfloor n}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)}x^{n-4} + \ldots\right]$$

which is $P_n(x)$

we find that $P_1(x), P_2(x)\ldots$ will be the coefficients of $z, z^2\ldots$ in the expansion of $(1-2xz+z^2)^{-1/2}$

$$(1-2zx+z^2)^{-1/2} = 1+zP_1(x)+z^2P_2(x)+\ldots+z^nP_n(x)$$

$$= \sum_{n=0}^{\infty} z^n P_n(x).$$

1.d) S.T $P_n(-x) = (-1)^n P_n(x)$ and hence that $P_n(-1) = (-1)^n$

Soln:

The generating function formula is

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x), \quad |x|\le 1, \quad |z|<1 \rightarrow ①$$

replacing x by $-x$ in ① we get,

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x) \rightarrow ②$$

Next let us replace z by $(-z)$ in ①

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x)$$

$$(1+2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x) \rightarrow (3)$$

from (2) and (3), we have,

$$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n P_n(x) \rightarrow (4)$$

equating the coefficient of z^n on both sides of (4)

we get,

$$P_n(-x) = (-1)^n P_n(x)$$

Putting $x=1$, $P_n(-1) = (-1)^n P_n(1)$ But $P_n(1)=1$

i.e) $P_n(-1) = (-1)^n$

1.e) S.T The coefficient of x^n in $P_n(x)$ is $\frac{\lfloor 2n}{2^n(\lfloor n)^2}$

Soln:

we have by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n} (x^2-1)^n$$

$$= \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n} [x^{2n} - nc_1 x^{2n-2} + nc_2 x^{2n-4} + \dots + (-1)^n]$$

$$= \frac{1}{2^n \lfloor n} [2n(2n-1)\dots(2n-\overline{n-1})x^n \dots 0]$$

∴ coefficient of x^n in $P_n(x)$

$$P_n(x) = \frac{1}{2^n \lfloor n} [2n.(2n-1)\dots(n+1)]$$

$$= \frac{1}{2^n \lfloor n} \frac{(2n(2n-1)\dots(n+1)n(n-1)\dots 3.2.1)}{(n(n-1)\dots 3.2.1)}$$

$$= \frac{\lfloor 2n}{2^n \lfloor n \lfloor n}$$

$$P_n(x) = \frac{\lfloor 2n}{2^n(\lfloor n)^2}$$

1.f) S.T $\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad (n \neq m)$

Soln:

Since $P_m(x)$ and $P_n(x)$ satisfy legendre eqn

we have,

$$(1-x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0 \rightarrow (1)$$

$$(1-x^2)P_m'' - 2xP_m' + m(m+1)P_m = 0 \rightarrow (2)$$

multiplying (1) by P_m and (2) by P_n

$$(1-x^2)P_n''P_m - 2xP_n'P_m + n(n+1)P_nP_m = 0 \rightarrow (3)$$

$$(1-x^2)P_m''P_n - 2xP_m'P_n + m(m+1)P_nP_m = 0 \rightarrow (4)$$

(4) - (3)

$$\left.\begin{aligned}(1-x^2)(P_nP_m'' - P_mP_n'') - 2x(P_nP_m' - P_n'P_m)\\ + [m(m+1) - n(n+1)]P_mP_n\end{aligned}\right\} = 0$$

i) $(1-x^2)\frac{d}{dx}(P_nP_m' - P_nP_m') - 2x(P_nP_m' - P_n'P_m) = (n^2 - m^2 + n - m)P_mP_n$

$$\frac{d}{dx}[(1-x^2)(P_nP_m' - P_n'P_m)] = (n-m)(n+m+1)P_mP_n$$

Integrating both sides w.r.t x from -1 to 1

$$(n-m)(n+m+1)\int_{-1}^{1} P_m(x)P_n(x)\, dx = [(1-x^2)(P_nP_m' - P_n'P_m)]_{-1}^{1}$$

$$\int_{-1}^{1} P_m(x) P_n(x)\, dx = 0 \quad \because \text{ as } m \neq n.$$

1.g) S.T $\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$

Soln:

$u(x) = (x^2-1)^n$ Then by Rodrigue's formula

$$P_n(x) = \frac{1}{2^n \lfloor n} u^{(n)}(x)$$

It can be S.T $u^{(k)}(1) = u^{(k)}(-1) = 0$ for $0 \leq k < n$

Then integrating by parts

$$\int_{-1}^{1} u^{(n)}(x)\, u^{(n)}(x)\, dx = \left[u^{(n)}(x)\, u^{(n-1)}(x)\right]_{-1}^{1} - \int_{-1}^{1} u^{(n+1)}(x)\, u^{(n-1)}(x)\, dx$$

$$= 0 - \int_{-1}^{1} u^{(n+1)}(x)\, u^{(n+1)}(x)\, dx$$

Continuing the process n times, we get,

$$\int_{-1}^{1} [u^n(x)]^2 dx = (-1)^n \int_{-1}^{1} u^{(2n)}(x)\, u(x)\, dx$$

But $u^{(2n)}(x) = \frac{d^{2n}}{dx^{2n}} (x^2-1)^n = \lfloor 2n \to ①$

$$\therefore \int_{-1}^{1} P_n^2(x)\,dx = \frac{1}{(2^n \lfloor n)^2} \int_{-1}^{1} [u^n(x)]^2 dx$$

$$= \frac{1}{(2^n \lfloor n)^2} \lfloor 2n \int_{-1} (-1)^n u(x)\, dx \quad \text{using ①}$$

$$= \frac{\lfloor 2n}{(2^n \lfloor n)^2} \int_{-1}^{1} (1-x^2)^n dx$$

Now, $\int_{-1}^{1} (1-x^2)^n dx = 2\int_0^{\pi/2} \cos^{2n+1}\phi\, d\phi$ where $x = \sin\phi$

$dx = \cos\phi\, d\phi$

$$= \frac{2(2^n \lfloor n)^2}{\lfloor 2n+1} \quad \text{using reduction formula}$$

$$\int_{-1}^{1} P_n^2(x)\,dx = \frac{\lfloor 2n}{(2^n \lfloor n)^2} \cdot \frac{2 \cdot 2^{2n} \lfloor n \lfloor n}{\lfloor 2n+1}$$

$$= \frac{2\lfloor 2n}{\lfloor 2n+1}$$

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}$$

Aliter:

We have

$$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$$

$$= P_0(x) + z P_1(x) + \dots + z^n P_n(x) + \dots$$

squaring both sides

$$(1-2xz+z^2)^{-1} = \sum_{n=0}^{\infty} z^{2n} [P_n(x)]^2 + \sum_{\substack{m,n=0 \\ m\neq n}}^{\infty} z^{m+n} P_m(x) P_n(x)$$

Integrating both sides w.r.t from -1 to 1

$$\int_{-1}^{1} \frac{dx}{1-2xz+z^2} = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{1} [P_n(x)]^2 dx + \sum_{\substack{m,n=0 \\ m\neq n}}^{\infty} z^{m+n} \int_{-1}^{1} P_m(x) P_n(x) dx = 0$$

$$\text{i.e.) } \left(\frac{\log(1-2xz+z^2)}{-2z}\right)_{-1}^{1} = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{1} [P_n(x)]^2 dx \quad \left[\because \int_{-1}^{1} P_m(x) P_n(x) dx = 0 \text{ if } m\neq n\right]$$

$$\frac{-1}{2z}\left[\log(1-z)^2 - \log(1+z)^2\right] = \sum_{n=0}^{\infty} z^{2n} \int_{-1}^{1} [P_n(x)]^2 dx$$

multiplying both sides by z and rearranging LHS $= -\frac{1}{2}\left[\log\frac{(1-z)^2}{(1+z)^2}\right] = -\log\left[\frac{(1-z)^2}{(1+z)^2}\right]^{1/2} = \log\frac{1+z}{1-z}$

$$\log\left(\frac{1+z}{1-z}\right) = \sum_{n=0}^{\infty} z^{2n+1} \int_{-1}^{1} [P_n(x)]^2 dx$$

$$2\left[z + \frac{z^3}{3} + \cdots + \frac{z^{2n+1}}{2n+1} + \cdots\right] = \sum_{n=0}^{\infty} z^{2n+1} \int_{-1}^{1} [P_n(x)]^2 dx$$

equating coefficient of z^{2n+1} on both sides

$$\frac{2}{2n+1} = \int_{-1}^{1} [P_n(x)]^2 dx$$

1. H) Let P be any polynomial of degree n and let $P = C_0P_0 + C_1P_1 + \cdots + C_nP_n$ where $C_0, C_1, \ldots C_n$ are constants s.t $C_K = \frac{2K+1}{2}\int_{-1}^{1} P(x) P_K \, dx \quad (K = 0, 1, 2 \ldots n)$

Soln:

Given

$$P(x) = C_0 P_0(x) + C_1 P_1(x) + \cdots + C_K P_K(x) + \cdots + C_n P_n(x)$$

multiplying both sides by $P_K(x)$ and integrating from -1 to 1

$$\int_{-1}^{1} P(x) P_K(x) dx = C_0 \int_{-1}^{1} P_0(x) P_K(x) dx + C_1 \int_{-1}^{1} P_1(x) P_K(x) dx + \cdots$$

$$+ C_K \int_{-1}^{1} P_K^2(x) dx + \cdots + C_n \int_{-1}^{1} P_n(x) P_K(x) dx$$

$\rightarrow (1)$

But $\int_{-1}^{1} P_m(x)\, P_n(x)\,dx = 0 \quad m \neq n$

and $\int_{-1}^{1} P_n^2(x)\,dx = \frac{2}{2n+1}$

These results sub in (1)

$$\int_{-1}^{1} P(x)\, P_K(x)\, dx = C_K \cdot \frac{2}{2K+1}$$

$$\therefore C_K = \frac{2K+1}{2} \int_{-1}^{1} P(x)\, P_K(x)\, dx, \quad (K = 0, 1, 2 \ldots)$$

1.i) Expand $f(x) = x^2$ in a series of the form $\Sigma C_r P_r(x)$ or find series of legendre polynomial for x^2

Soln:

$$x^2 = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x)$$

where $C_r = \left(r + \frac{1}{2}\right) \int_{-1}^{1} x^2 P_r(x)\, dx$

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

Put $r = 0$.

$$\therefore C_0 = \frac{1}{2} \int_{-1}^{1} x^2\, dx = \left(\frac{x^3}{6}\right)_{-1}^{1}$$

$$C_0 = \frac{1}{3}$$

Put $r = 1$, $C_1 = \frac{3}{2} \int_{-1}^{1} x^3\, dx = 0$

$r = 2$, $C_2 = \frac{5}{2} \cdot \frac{1}{2} \int_{-1}^{1} x^2 (3x^2 - 1)\, dx = \frac{2}{3}$

Sub the values of C_0, C_1, C_2 we get.

$$x^2 = \frac{1}{3} P_0(x) + \frac{2}{3} P_1(x)$$

[Note: This problem can be done using the method of 1(a) problem]

But $\int_{-1}^{1} P_m(x) P_n(x) dx = 0 \quad m \neq n$

and $\int_{-1}^{1} P_n^2(x) dx = \frac{2}{2n+1}$

These results sub in (1)

$$\int_{-1}^{1} P(x) P_k(x) dx = C_k \cdot \frac{2}{2k+1}$$

$$\therefore C_k = \frac{2k+1}{2} \int_{-1}^{1} P(x) P_k(x) dx, \quad (k = 0, 1, 2 \dots)$$

1.i) Expand $f(x) = x^2$ in a series of the form $\sum C_r P_r(x)$ (or)

find series of legendre polynomial for x^2

Soln:

$$x^2 = C_0 P_0(x) + C_1 P_1(x) + C_2 P_2(x)$$

where $C_r = (r + \frac{1}{2}) \int_{-1}^{1} x^2 P_r(x) dx$

$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1)$

Put $r = 0$.

$$\therefore C_0 = \frac{1}{2} \int_{-1}^{1} x^2 dx = \left(\frac{x^3}{6}\right)_{-1}^{1}$$

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[Note: This problem can be done using the method of 1(a) problem]